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Some Conver Gence Resulats for Fixed Points of Asymptotically Demicontractive Mappings in Some Banach Spaces

*** Anil Rajput ** S.K. Malhotra**
***** Pradeep Kumar**

* Department of Mathematics, Sadhu Vaswani College, Bairagarh, Bhopal.
** Department of Mathematics, Govt.S.G.S.P.G.College, Ganjbasoda.
*** Department of Mathematics, Sadhu Vaswani College, Bairagarh, Bhopal.

A B S T R A C T

In this paper we establish some convergence results for the improved iteration methods introduced by Owojori and Imoru [7] to fixed points of asymptotically demi-contractive mappings. Our results in this paper are extension of Olusegun O.Owojori [8] and also generalization of Ishikawa [4], Deng and Ding [3], Chidume [1], Chidume and Osilike [2], Owojori and Imoru [6], Qihou [9], Liu [5], and Xu [10] from the Mann and Ishikawa iteration methods to more general iteration method and from Lipschitz pseudocontractive mapping to more general continuous demi-contractive mapping.

Introduction: Xu [10], introduced suitable Mann and Ishikawa iteration scheme with errors for approximation of fixed points and solutions of nonlinear mappings in Banach spaces. Owojori and Imoru [6] introduced a three step iteration scheme and obtained some convergence results to the fixed points of continuous asymptotically demi-contractive mapping in Hilbert spaces. Owojori and Imoru [7] introduced and improved three step iteration method which contains the one introduced earlier by the authors in [6] as well as the Mann and Ishikawa iteration methods as special cases. It is defined for arbitrary $x_1 \in K$ a closed bounded convex subset of a Banach space B, by

$$x_{n+1} = a_n x_n + b_n T^n x_n + c_n S^n x_n \quad n = 1$$
$$y_n = a_n' x_n + b_n' S^n z_n + c_n' v_n \quad n = 1$$
$$z_n = a_n'' x_n + b_n'' T^n x_n + c_n'' w_n \quad n = 1 \qquad \ldots(1.1)$$

Two special cases of (1.1) are given respectively by

$$x_{n+1} = a_n x_n + b_n T^n y_n + c_n T^n x_n \quad n = 1$$
$$y_n = a_n' x_n + b_n' T^n z_n + c_n' v_n \quad n = 1$$
$$z_n = a_n'' x_n + b_n'' T^n x_n + c_n'' w_n \quad n = 1 \quad \ldots(1.2)$$

and

$$x_{n+1} = a_n x_n + b_n T^n y_n + c_n u_n \quad n = 1$$
$$y_n = a_n' x_n + b_n' T^n z_n + c_n' v_n \quad n = 1$$
$$z_n = a_n'' x_n + b_n'' T^n x_n + c_n'' w_n \quad n = 1 \qquad \ldots(1.3)$$

were S, T are nonlinear uniformly continuous self mapping of K satisfying some contractive definitions and $\{u_n\},\{v_n\},\{w_n\}$ are bounded sequence in K. Also $\{a_n\},\{a_n'\}\{a_n''\},\{b_n\},\{b_n'\},\{b_n''\},\{c_n\},\{c_n'\},\{c_n''\}$ are real sequence in [0,1] satisfying:

(i) $a_n + b_n + c_n = a_n' + b_n' + c_n' = a_n'' + b_n'' + c_n'' = 1$ (ii) $\sum b_n = \infty$.

Let H be a Hilbert space and let K be nonempty subset of H. A mapping $T : K \to K$ is said to be k-strictly asymptotically pseudo-contractive if there exists a sequence $\{k_n\}$ with $\lim_{n\to\infty} k_n = 1$ such that:

$\| T^n x - T^n y \|^2 \le k_n^2 \|x - y\|^2 + k \|(I - T^n)x - (I - T^n)y\|^2$ …(1.4) for some $k \in [0,1)$ and for all $x,y \in K$ and $n \in N$. $T : K \to K$ is said to be T is asymptotically demi-contractive if, $F(T) = \{x \in K : Tx = x\} \neq 0$ and there exists a sequence $\{k_n\}$ with $\lim_{n\to\infty} k_n = 1$ such that: $\| T^n x - x^* \|^2 \le k_n^2 \| x - x^* \|^2 + k \|x - T^n x\|^2$ …(1.5) from some $k \in [0,1)$ and for all $x \in K$, $x^* \in F\{T\}$

and $n \in N$. Our purpose in this paper is to establish the convergence of the iteration methods (1.3),(1.2) and (1.1) to the fixed points and common fixed points of the asymptotically demi-contractive mapping in arbitrary Hilbert spaces. Our purpose in this paper generalize the results of Olusegun O. Owojori [8].

Main Results: Lemma 2.1: Let B be a uniformly smooth Banach space with modulus of smoothness of power type the following inequality $\|\lambda x + (1-\lambda)y - z\|^q \le [1-\lambda(q-1)]\|y-z\|^q + \lambda c\|x-z\|^q - \lambda[1-\lambda^{q-1}c]\|x-y\|^q$ ...(2.1) holds where c is a positive constant.

Remark: It is known that for a Hilbert space(which is a special Banach space) q = 2 and c = 1.Therefore in a Hilbert space H say (2.1) reduces to $\|\lambda x + (1-\lambda)y - z\|^2 \le [1-\lambda]\|y-z\|^2 + \lambda\|x-z\|^2 - \lambda[1-\lambda]\|x-y\|^2$ ...(2.2) for all $x,y,z \in H$ and $\lambda \in [0, 1]$ Lemma 2.2: Let $\{\phi_n\}$ be a nonnegative sequence of real number satisfying $\phi_{n+1} \le (1-\delta n)\phi_n + \sigma_n$ ...(2.3)

where $\delta n \in [0, 1]$, $\sum \delta i = \infty$, and $\sigma_n = o(\delta n)$. Then $\lim_{n\to\infty} \phi_n = 1$.Our results is the following.

Theorem 2.3: Let B be an arbitrary Hilbert space and K be a nonempty closed bounded and convex subset of B. Suppose T is a continuous asymptotically demi-contractive self mapping of K. Define sequence $\{x_n\}$ iteratively for arbitrary $x_1 \in K$ by:

$x_{n+1} = a_n x_n + b_n T^n y_n + c_n u_n$ n = 1

$y_n = a_n' x_n + b_n' T^n z_n + c_n' v_n$ n = 1

$z_n = a_n'' x_n + b_n'' T^n x_n + c_n'' w_n$ n = 1

where $\{u_n\},\{v_n\}$and $\{w_n\}$ are bounded sequence in K and $\{a_n\},\{a_n'\},\{a_n''\},\{b_n\}, \{b_n'\},\{b_n''\},\{c_n\}, \{c_n'\},\{c_n''\}$ are real sequence in [0,1] satisfying:

(1) $a_n + b_n + c_n = a_n' + b_n' + c_n' = a_n'' + b_n'' + c_n'' = 1$ (2) $\sum b_n' = \infty$ (3) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b_n' = \lim_{n\to\infty} b_n'' = 0$ (4) $\alpha_n := b_n + c_n, \beta_n := b_n' + c_n', \gamma_n := b_n'' + c_n''$ (5) $\sum \alpha_n \beta_n \gamma_n = \infty$ Then the sequence $\{x_n\}$ convergence strongly to the fixed point of T.

Proof: Since T is asymptotically demi-contractive then F(T) the fixed point set of T is nonempty. Let $x^* \in F(T)$. From the hypothesis we have, $\|x_{n+1} - x^*\|^2 = \|a_n x_n + b_n T^n y_n + c_n u_n - x^*\|^2 = \|(1-\alpha_n)(x_n - x^*) + \alpha_n(T^n y_n - x^*) - c_n(T^n y_n - u_n)\|^2 \le (1-\alpha_n)\|(x_n - x^*) - c_n(T^n y_n - u_n)\|^2 + \alpha_n\|(T^n y_n - x^*) - c_n(T^n y_n - u_n)\| - \alpha_n(1-\alpha_n)\|(T^n y_n - x^*) - (x_n - x^*)\|^2$ Since $\alpha_n(1-\alpha_n) \ge 0$, then we have $\|x_{n+1} - x^*\|^2 \le (1-\alpha_n)\|(x_n - x^*) - c_n(T^n y_n - u_n)\|^2 + \alpha_n\|(T^n y_n - x^*) - c_n(T^n y_n - u_n)\|^2$ Expanding further and observing that $\|a-b\|^2 \le \|a\|^2 + \|b\|^2$, where a, b are real numbers we obtain $\|x_{n+1} - x^*\|^2 \le (1-\alpha_n)[\|x_n - x^*\|^2 + c_n^2\|T^n y_n - u_n\|^2] + \alpha_n[\|T^n y_n - x^*\|^2 + c_n^2\|T^n y_n - u_n\|^2] - 2c_n\langle T^n y_n - u_n, j(T^n y_n - x^*)\rangle$

$\|x_{n+1} - x^*\|^2 \le (1-\alpha_n)[\|x_n - x^*\|^2 + c_n^2\|T^n y_n - u_n\|^2] + \alpha_n[\|T^n y_n - x^*\|^2 + c_n^2\|T^n y_n - u_n\|^2] \le (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|T^n y_n - x^*\|^2 + c_n^2\|T^n y_n - u_n\|^2$ ...(2.4)

Since T is asymptotically demi-contractive then

$\|T^n y_n - x^*\|^2 \le k_n^2\|y_n - x^*\|^2 + k\|y_n - T^n y_n\|^2$ ...(2.5) we have the following estimates: $\|y_n - x^*\|^2 = \|a_n' x_n + b_n' T^n z_n + c_n' v_n - x^*\|^2 = \|(1-\beta_n)(x_n - x^*) + \beta_n(T^n z_n - x^*) - c_n'(T^n z_n - v_n)\|^2 \le (1-\beta_n)\|(x_n - x^*) - c_n'(T^n z_n - v_n)\|^2 + \beta_n\|(T^n z_n - x^*) - c_n'(T^n z_n - v_n)\|^2 - \beta_n(1-\beta_n)\|(T^n z_n - x^*) - (x_n - x^*)\|^2$

Observe that $\beta_n(1-\beta_n) \ge 0$ therefore $\|y_n - x^*\|^2 \le (1-\beta_n)[\|x_n - x^*\|^2 + \beta_n^2\|T^n z_n - v_n\|^2] + \beta_n[\|T^n z_n - x^*\|^2 + \beta_n^2\|T^n z_n - v_n\|^2] = (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|T^n z_n - x^*\|^2 + \beta_n^2\|T^n z_n - v_n\|^2$ ...(2.6)

T is asymptotically demi-contractive therefore

$\|T^n z_n - x^*\|^2 \le k_n^2\|z_n - x^*\|^2 + k\|z_n - T^n z_n\|^2$ ...(2.7) Substituting (2.7) into (2.6) yields $\|y_n - x^*\|^2 \le (1-\beta_n)\|x_n - x^*\|^2 + \beta_n k_n^2\|z_n - x^*\|^2 + \beta_n^2\|T^n z_n - v_n\|^2 + \beta_n k\|z_n - T^n z_n\|^2$...(2.8)

Substituting (2.8) into (2.5)

$\|T^n y_n - x^*\|^2 \le k_n^2[(1-\beta_n)\|x_n - x^*\|^2 + \beta_n k_n^2\|z_n - x^*\|^2 + \beta_n^2\|T^n z_n - v_n\|^2 + \beta_n k\|z_n - T^n z_n\|^2] + k\|y_n - T^n y_n\|^2$...(2.9)

Further estimates gives the following $\|z_n - x^*\|^2 = \|a_n'' x_n + b_n'' T^n x_n + c_n'' w_n - x^*\|^2 = \|(1-\gamma_n)(x_n - x^*) + \gamma_n(T^n x_n - x^*) - c_n''(T^n x_n - w_n)\|^2 \le (1-\gamma_n)\|(x_n - x^*) - c_n''(T^n x_n - w_n)\|^2 + \gamma_n\|(T^n x_n - x^*) - c_n''(T^n x_n - w_n)\|^2 \le (1-\gamma_n)\|x_n - x^*\|^2 + \gamma_n\|T^n x_n - x^*\|^2 + \gamma_n^2(1-\gamma_n)\|T^n x_n - w_n\|^2 + \gamma_n^2\gamma_n\|T^n x_n - w_n\|^2 \le (1-\gamma_n)\|x_n - x^*\|^2 + \gamma_n\|T^n x_n - x^*\|^2 + \gamma_n^2\|T^n x_n - w_n\|^2$ ...(2.10)

By continuity of T and bounded ness on K there exists a real numbers $M_1 < \infty$ such that:

$\|T^n x_n - x^*\|^2 \le M_1$ and $\|T^n x_n - w_n\|^2 \le M_1$. Observe that $c_n'' < \gamma_n$ and $\gamma_n^2 < \gamma_n$ for all n. Then (2.10) reduces to $\|z_n - x^*\|^2 \le (1-\gamma_n)\|x_n - x^*\|^2 + 2\gamma_n M_1$ ...(2.11)

We now substitute (2.11) into (2.9) to get $\|T^n y_n - x^*\|^2 \le k_n^2[(1-\beta_n)\|x_n - x^*\|^2 + \beta_n k_n^2\{(1-\gamma_n)\|x_n - x^*\|^2 + 2\gamma_n M_1\} + \beta_n^2\|T^n z_n - v_n\|^2 + \beta_n k\|z_n - T^n z_n\|^2] + k\|y_n - T^n y_n\|^2 \le k_n^2[(1-\beta_n + \beta_n k_n^2 - \beta_n k_n^2\gamma_n)\|x_n - x^*\|^2 + 2\beta_n k_n^2\gamma_n M_1 + \beta_n^2\|T^n z_n - v_n\|^2 + \beta_n k\|z_n - T^n z_n\|^2] + k$

$\|y_n - T^n y_n\|^2 \ldots(2.12)$

By continuity of T on the bounded set K there exists a real number $M_2 < \infty$ such that: $\| z_n - T^n z_n \|^2 \le M_2$, $\|y_n - T^n y_n\|^2 \le M_2$ and $\|T^n z_n - v_n\|^2 \le M_2$ Therefore (2.12) reduces to $\|T^n y_n - x^*\|^2 \le k_n{}^2[(1-\beta_n+\beta_n k_n{}^2 - \beta_n k_n{}^2\gamma_n)\|x_n - x^*\|^2 + 2\beta_n k_n{}^2\gamma_n M_1 + \beta_n{}^2 M_2 + \beta_n k M_2] + k M_2 \le k_n{}^2(1-\beta_n+\beta_n k_n{}^2 - \beta_n k_n{}^2\gamma_n)\|x_n - x^*\|^2 + 2\beta_n k_n{}^4\gamma_n M_1 + \beta_n{}^2 k_n{}^2 M_2 + \beta_n k_n{}^2 k M_2 + k M_2 \le k_n{}^2(1-\beta_n + \beta_n k_n{}^2 - \beta_n k_n{}^2\gamma_n)\|x_n - x^*\|^2 + 2\beta_n k_n{}^4\gamma_n M_1 + (\beta_n{}^2 k_n{}^2 + \beta_n k_n{}^2 k + k)M_2 \ldots(2.13)$ substitute (2.13) into (2.4) we have

$\|x_{n+1} - x^*\|^2 \le (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n[k_n{}^2(1-\beta_n+\beta_n k_n{}^2 - \beta_n k_n{}^2\gamma_n)\|x_n - x^*\|^2 + 2\beta_n k_n{}^4\gamma_n M_1 + (\beta_n{}^2 k_n{}^2 + \beta_n k_n{}^2 k + k)M_2] + c_n{}^2\|T^n y_n - u_n\|^2 \le [1-\alpha_n+\alpha_n k_n{}^2 - \alpha_n\beta_n k_n{}^2(1-k_n{}^2+k_n{}^2\gamma_n)]\|x_n - x^*\|^2 + 2\alpha_n\beta_n\gamma_n k_n{}^4 M_1 + \alpha_n(\beta_n{}^2 k_n{}^2 + \beta_n k_n{}^2 k + k)M_2 + c_n{}^2\|T^n y_n - u_n\|^2 \ldots(2.14)$ Continuity of T on K also implies that there exists real number $M_3 < \infty$ such that:

$\|T^n y_n - u_n\|^2 \le M_3$. Let $M_6 = \max[M_1, M_2, M_3]$. Therefore from (2.14) and the fact that $c_n{}^2 < \alpha_n{}^2 < \alpha_n$ we have $\|x_{n+1} - x^*\|^2 \le [1-\alpha_n+\alpha_n k_n{}^2 - \alpha_n\beta_n k_n{}^2(1-k_n{}^2+k_n{}^2\gamma_n)]\|x_n - x^*\|^2 + [2\alpha_n\beta_n\gamma_n k_n{}^4 + \alpha_n(\beta_n{}^2 k_n{}^2 + \beta_n k_n{}^2 k + k) + \alpha_n]M_6 \ldots(2.15)$

Now put $\phi_n = \|x_n - x^*\|^2$, $\delta_n = \alpha_n + \alpha_n k_n{}^2 - \alpha_n\beta_n k_n{}^2(1-k_n{}^2+k_n{}^2\gamma_n)$ and $\sigma_n = [2\alpha_n\beta_n\gamma_n k_n{}^4 + \alpha_n(\beta_n{}^2 k_n{}^2 + \beta_n k_n{}^2 k + k) + \alpha_n]M_6$

Then (2.15) becomes; $\phi_{n+1} \le (1-\delta_n)\phi_n + \sigma_n$, $n \ge 1$. Clearly, $\sigma_n = o(\delta_n)$, $\sum\delta_n = \infty$ and $0 \le \delta_n \le 1$. Hence $\phi_{n \to \infty}$ as $n \to \infty$. This implies that $\{x_n\}$ converges strongly to x*. This completes the proof.

Theorem 2.4: Let K be a closed bounded convex nonempty subset of a Hilbert space H. Let T be a completely continuous asymptotically demi-contractive self mapping of K. Define a sequence $\{x_n\}$ iteratively for arbitrary $x_n \in K$ by:

$x_{n+1} = a_n x_n + b_n T^n y_n + c_n T^n x_n \quad n = 1$

$y_n = a_n' x_n + b_n' T^n z_n + c_n' v_n \quad n = 1$

$z_n = a_n'' x_n + b_n'' T^n x_n + c_n'' w_n \quad n = 1$

where $\{v_n\}$ and $\{w_n\}$ are bounded sequence in K and $\{a_n\},\{a_n'\},\{a_n''\},\{b_n\},\{b_n'\},\{b_n''\},\{c_n\},\{c_n'\},\{c_n''\}$ are real sequence in [0,1] satisfying the following condition:

(1) $a_n + b_n + c_n = a_n' + b_n' + c_n' = a_n'' + b_n'' + c_n'' = 1$ (2) $\sum b_n = \infty$ (3) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b_n' = \lim_{n\to\infty} b_n'' = 0$ (4) $\alpha_n = b_n + c_n$, $\beta_n = b_n' + c_n'$, $\gamma_n = b_n'' + c_n''$ (5) $\sum \alpha_n \beta_n \gamma_n = \infty$ Then the sequence $\{x_n\}$ converges strongly to the fixed point of T.

Proof: T is asymptotically demi-contractive implies that F(T) the fixed point set of T is nonempty. Let $x^* \in F(T)$. Then from our hypothesis we have the following estimates:

$\|x_{n+1} - x^*\|^2 = \|a_n x_n + b_n T^n y_n + c_n T^n x_n - x^*\|^2 = \|(1-\alpha_n)(x_n - x^*) + \alpha_n(T^n y_n - x^*) - c_n(T^n y_n - T^n x_n)\|^2 \le (1-\alpha_n)\|(x_n - x^*) - c_n(T^n y_n - T^n x_n)\|^2 + \alpha_n\|(T^n y_n - x^*) - c_n(T^n y_n - T^n x_n)\|^2 - \alpha_n(1-\alpha_n)\|(T^n y_n - x^*) - (x_n - x^*)\|^2$ Expanding further and observing that $\alpha_n(1-\alpha_n) \ge 0$, we have $\|x_{n+1} - x^*\|^2 \le (1-\alpha_n)[\|x_n - x^*\|^2 + c_n{}^2\|T^n y_n - T^n x_n\|^2] + \alpha_n[\|T^n y_n - x^*\|^2 + c_n{}^2\|T^n y_n - T^n x_n\|^2] \le (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|T^n y_n - x^*\|^2 + c_n{}^2\|T^n y_n - T^n x_n\|^2$ Since T is asymptotically demi-contractive and $c_n{}^2 \le \alpha_n{}^2 \le \alpha_n$ we have $\|x_{n+1} - x^*\|^2 \le (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n k_n{}^2\|y_n - x^*\|^2 + \alpha_n k\|y_n - T^n y_n\|^2 + \alpha_n\|T^n y_n - T^n x_n\|^2 \ldots(2.16)$

T is completely continuous on the bounded set K implies that there exists a real number $M_4 < \infty$ such that: $\|T^n y_n - T^n x_n\|^2 \le M_4$ and $\|y_n - T^n y_n\|^2 \le M_4$. Substituting into (2.16) yields $\|x_{n+1} - x^*\|^2 \le (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n k_n{}^2\|y_n - x^*\|^2 + \alpha_n k M_4 + \alpha_n M_4 \ldots(2.17)$

From our hypothesis we also have the following

$\|y_n - x^*\|^2 = \|a_n' x_n + b_n' T^n z_n + c_n' v_n - x^*\|^2 = \|(1-\beta_n)(x_n - x^*) + \beta_n(T^n z_n - x^*) - c_n'(T^n z_n - v_n)\|^2 \le (1-\beta_n)\|(x_n - x^*) - c_n'(T^n z_n - v_n)\|^2 + \beta_n\|(T^n z_n - x^*) - c_n'(T^n z_n - v_n)\|^2 - \beta_n(1-\beta_n)\|(T^n z_n - x^*) - (x_n - x^*)\|^2 \le (1-\beta_n)[\|x_n - x^*\|^2 + \beta_n{}^2\|T^n z_n - v_n\|^2] + \beta_n[\|T^n z_n - x^*\|^2 + \beta_n{}^2\|T^n z_n - v_n\|^2] = (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|T^n z_n - x^*\|^2 + \beta_n{}^2\|T^n z_n - v_n\|^2 \le (1-\beta_n)\|x_n - x^*\|^2 + \beta_n k_n{}^2\|z_n - x^*\|^2 + \beta_n k\|z_n - T^n z_n\|^2 + \beta_n{}^2\|T^n z_n - v_n\|^2 \ldots(2.18)$

Continuity of T on K implies that there exists a real number $M_5 < \infty$ such that: $\|z_n - T^n z_n\|^2 \le M_5$ and $\|T^n z_n - v_n\|^2 \le M_5$. Then (2.18) reduces to: $\|y_n - x^*\|^2 \le (1-\beta_n)\|x_n - x^*\|^2 + \beta_n k_n{}^2\|z_n - x^*\|^2 + \beta_n k M_5 + \beta_n{}^2 M_5 \ldots(2.19)$

By similar estimates we have $\|z_n - x^*\|^2 \le (1-\gamma_n)\|x_n - x^*\|^2 + \gamma_n k_n{}^2\|x_n - x^*\|^2 + \gamma_n k M_7 + \gamma_n{}^2 M_7 \ldots(2.20)$

for some real number $M_7 < \infty$. Substituting (2.20) into (2.19) yields: $\|y_n - x^*\|^2 \le (1-\beta_n)\|x_n - x^*\|^2 + \beta_n k_n{}^2[(1-\gamma_n)\|x_n - x^*\|^2 + \gamma_n k_n{}^2\|x_n - x^*\|^2 + \gamma_n k M_7 + \gamma_n{}^2 M_7] + \beta_n k M_5 + \beta_n{}^2 M_5 \le [(1-\beta_n) + \beta_n k_n{}^2(1-\gamma_n) + \beta_n\gamma_n k_n{}^4]\|x_n - x^*\|^2 + \beta_n(\gamma_n k_n{}^2 k + \gamma_n{}^2 k_n{}^2 + k + \beta_n)M_8 \ldots(2.21)$ where $M_8 = \max[M_5, M_7]$ substitute (2.21) into

(2.17) we obtain $\|x_{n+1} - x^*\|^2 \le (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n k_n^2[\{(1-\beta_n)+\beta_n k_n^2(1-\gamma_n)+\beta_n\gamma_n k_n^4\}\|x_n - x^*\|^2 + \beta_n(\gamma_n k_n^2 k + \gamma_n^2 k_n^2 + k + \beta_n) M_8] + \alpha_n k M_4 + \alpha_n M_4 \le (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n k_n^2\{(1-\beta_n)+\beta_n k_n^2(1-\gamma_n)+\beta_n\gamma_n k_n^4\}\|x_n-x^*\|^2 + \alpha_n\beta_n k_n^2(\gamma_n k_n^2 k + \gamma_n^2 k_n^2 + k + \beta_n)M_8 + \alpha_n k M_4 + \alpha_n M_4$.

Let $M_9 = \max[M_4, M_8]$. Then we have $\|x_{n+1}-x^*\|^2 \le [1-\alpha_n+\alpha_n k_n^2 - \alpha_n\beta_n k_n^2 + \alpha_n\beta_n k_n^4 - \alpha_n\beta_n\gamma_n k_n^4 + \alpha_n\beta_n\gamma_n k_n^6]\|x_n-x^*\|^2 + \alpha_n[\beta_n\gamma_n k_n^4 k + \beta_n\gamma_n^2 k_n^4 + \beta_n k_n^2 k k_n^2 + k + 1]M_9$ ...(2.22)

Now put $\phi_n = \|x_n - x^*\|^2$ and $\delta_n = \alpha_n + \alpha_n k_n^2 - \alpha_n\beta_n k_n^2 + \alpha_n\beta_n k_n^4 - \alpha_n\beta_n\gamma_n k_n^4 + \alpha_n\beta_n\gamma_n k_n^6$ Also let $\sigma_n = \alpha_n[\beta_n\gamma_n k_n^4 k + \beta_n\gamma_n^2 k_n^4 + \beta_n k_n^2 k + \beta_n^2 k_n^2 + k + 1]M_9$

Then (2.22) reduces to: $\phi_{n+1} = (1-\delta_n)\phi_n + \sigma_n$ observe that $0 \le \delta_n \le 1$, $\sum\delta_n = \infty$ and $\sigma_n = o(\delta_n)$.

Therefore by Lemma 2.2 $\phi_{n\to 0}$ as $n\to\infty$. This implies that the sequence $\{x_n\}$ converges strongly to x*. This complete the proof.

Theorem 2.5: Let K be a nonempty closed bounded convex subset of a Hilbert space H. Suppose S,T are uniformly continuous self mapping of K and T is asymptotically demi-contractive on K. Define a sequence $\{x_n\}$ iteratively for arbitrary $x_n \in K$ by:

$x_{n+1} = a_n x_n + b_n T^n y_n + c_n S x_n$ n = 1 $y_n = a_n' x_n + b_n' S z_n + c_n' v_n$ n = 1 $z_n = a_n'' x_n + b_n'' T^n x_n + c_n'' w_n$ n =1

where $\{v_n\}$ and $\{w_n\}$ are bounded sequence in K and $\{a_n\},\{a_n'\},\{a_n''\},\{b_n\},\{b_n'\},\{b_n''\},\{c_n\},\{c_n'\},\{c_n''\}$ are real sequence in [0,1] satisfying the following condition:

(1) $a_n + b_n + c_n = a_n' + b_n' + c_n' = a_n'' + b_n'' + c_n'' = 1$ (2) $\sum b_n = \infty$ (3) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b_n' = \lim_{n\to\infty} b_n'' = 0$ (4) $\alpha_n = b_n + c_n$, $\beta_n = b_n' + c_n'$, $\gamma_n = b_n'' + c_n''$ (5) $\sum \alpha_n\beta_n\gamma_n = \infty$ Then the sequence $\{x_n\}$ converges strongly to the fixed point of T.

Proof: Since T is asymptotically demi-contractive then F(T) is nonempty. Let $x^* \in F(T)$. Then from our hypothesis we have:

$\|x_{n+1} - x^*\|^2 = \|a_n x_n + b_n T^n y_n + c_n S x_n - x^*\|^2 = \|(1-\alpha_n)(x_n - x^*) + \alpha_n(T^n y_n - x^*) - c_n(T^n y_n - S x_n)\|^2$ $\|x_{n+1} - x^*\|^2 \le (1-\alpha_n)\|(x_n - x^*) - c_n(T^n y_n - S x_n)\|^2 + \alpha_n\|(T^n y_n - x^*) - c_n(T^n y_n - S x_n)\|^2 - \alpha_n(1-\alpha_n)\|(T^n y_n - x^*) - (x_n - x^*)\|^2 \le (1-\alpha_n)[\|x_n - x^*\|^2 + c_n^2\|T^n y_n - S x_n\|^2] + \alpha_n[\|T^n y_n - x^*\|^2 + c_n^2\|T^n y_n - S x_n\|^2] = (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|T^n y_n - x^*\|^2 + c_n^2\|T^n y_n - S x_n\|^2 = (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n k_n^2\|y_n - x^*\|^2 + \alpha_n k\|y_n - T^n y_n\|^2 + \alpha_n\|T^n y_n - S x_n\|^2$...(2.23)

Continuity of S,T implies that there exists real number $q_1, q_2 < \infty$ such that: $\|y_n - T^n y_n\|^2 \le q_1$ and $\|T^n y_n - S x_n\|^2 \le q_2$. Let $q_3 = \max[q_1, q_2]$. Then (2.23) yields: $\|x_{n+1} - x^*\|^2 = (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n k_n^2\|y_n - x^*\|^2 + \alpha_n k q_3 + \alpha_n q_3$ ...(2.24)

We also have the following estimates

$\|y_n - x^*\|^2 = \|a_n' x_n + b_n' S z_n + c_n' v_n - x^*\|^2 = \|(1-\beta_n)(x_n - x^*) + \beta_n(S z_n - x^*) - c_n'(S z_n - v_n)\|^2 \le (1-\beta_n)\|(x_n - x^*) - c_n'(S z_n - v_n)\|^2 + \beta_n\|(S z_n - x^*) - c_n'(S z_n - v_n)\|^2 - \beta_n(1-\beta_n)\|(S z_n - x^*) - (x_n - x^*)\|^2 \le (1-\beta_n)[\|x_n - x^*\|^2 + \beta_n^2\|S z_n - v_n\|^2] + \beta_n[\|S z_n - x^*\|^2 + \beta_n^2\|S z_n - v_n\|^2] = (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|S z_n - v_n\|^2 + \beta_n\|S z_n - x^*\|^2$ ...(2.25)

Continuity of S on K implies there exists a real number $q_4 < \infty$ such that:

$\|S z_n - x^*\|^2 \le q_4$ and $\|S z_n - v_n\|^2 \le q_4$. Then from (2.25) we have: $\|y_n - x^*\|^2 \le (1-\beta_n)\|x_n - x^*\|^2 + 2\beta_n q_4$ ...(2.26) Situting (2.26) into (2.24) we have

$\|x_{n+1} - x^*\|^2 = (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n k_n^2[(1-\beta_n)\|x_n - x^*\|^2 + 2\beta_n q_4] + \alpha_n k q_3 + \alpha_n q_3$ ...(2.27)

Let $q_5 = \max[q_3, q_4]$. Then $\|x_{n+1} - x^*\|^2 = \{(1-\alpha_n) + \alpha_n k_n^2(1-\beta_n)\}\|x_n - x^*\|^2 + 2\alpha_n\beta_n k_n^2 q_5 + \alpha_n k q_5 + \alpha_n q_5$ ...(2.28) Putting $\rho_n = \|x_n - x^*\|^2$ and $\delta_n = \alpha_n + \alpha_n k_n^2 - \alpha_n\beta_n k_n^2$ $\sigma_n = 2\alpha_n\beta_n k_n^2 q_5 + \alpha_n k q_5 + \alpha_n q_5$ Then (2.28) reduces to: $\rho_{n+1} = (1-\delta_n)\rho_n + \sigma_n$.

Clearly $0 \le \delta_n \le 1$, $\sum\delta_n = \infty$ and $\sigma_n = o(\delta_n)$. Hence by Lemma 2.2 $\rho_{n\to 0}$ as $n\to\infty$. This implies that the sequence $\{x_n\}$ converges strongly to x*. This complete the proof.

Theorem 2.6: Let K be a nonempty closed bounded convex subset of a Hilbert space H. Suppose S,T are uniformly continuous asymptotically demi-contractive self mapping of K. Define a sequence $\{x_n\}$ iteratively for arbitrary $x_n \in K$ by:

$x_{n+1} = a_n x_n + b_n T^n y_n + c_n S^n x_n$ n = 1

$y_n = a_n' x_n + b_n' S^n z_n + c_n' v_n$ n = 1

$z_n = a_n'' x_n + b_n'' T^n x_n + c_n'' w_n$ n = 1

where $\{v_n\}$ and $\{w_n\}$ are bounded sequence in K and $\{a_n\},\{a_n'\},\{a_n''\},\{b_n\},\{b_n'\},\{b_n''\},\{c_n\},\{c_n'\},\{c_n''\}$ are real sequence in [0,1] satisfying the following condition:

(1) $a_n + b_n + c_n = a_n' + b_n' + c_n' = a_n'' + b_n'' + c_n'' = 1$(2) $\sum b_n = \infty$(3) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b_n' = \lim_{n\to\infty} b_n'' = 0$(4) $\alpha_n = b_n + c_n$, $\beta_n = b_n' + c_n'$, $\gamma_n = b_n'' + c_n''$(5) $\sum \alpha_n\beta_n\gamma_n = \infty$. If S,T have a common fixed point in K then the sequence$\{x_n\}$converge strongly to the common

fixed point of S and T.

Proof: Since S,T are asymptotically demi-contractive then the fixed point sets F(S) and F(T) are nonempty. Let p be a common fixed point of S and T. By our hypothesis and lemma 2.2 we have the following estimates:

$\|x_{n+1} - p\|^2 = \|a_n x_n + b_n T^n y_n + c_n S^n x_n - p\|^2 = \|(1-\alpha_n)(x_n - p) + \alpha_n (T^n y_n - p) - c_n (T^n y_n - S^n x_n)\|^2 \le (1-\alpha_n)\|(x_n - p) - c_n (T^n y_n - S^n x_n)\|^2 + \alpha_n \|(T^n y_n - p) - c_n (T^n y_n - S^n x_n)\|^2 - \alpha_n (1-\alpha_n)\|(T^n y_n - p) - (x_n - p)\|^2$ Observe that $\alpha_n (1-\alpha_n) \ge 0$ and $c_n^2 \le \alpha_n^2 \le \alpha_n$. Therefore expanding the above further yields $\|x_{n+1} - p\|^2 \le (1-\alpha_n)\|(x_n - p) - c_n (T^n y_n - S^n x_n)\|^2 + \alpha_n \|(T^n y_n - p) - c_n (T^n y_n - S^n x_n)\|^2 \le (1-\alpha_n)[\|x_n - p\|^2 + c_n^2 \|T^n y_n - S^n x_n\|^2] + \alpha_n [\|T^n y_n - p\|^2 + c_n^2 \|T^n y_n - S^n x_n\|^2] = (1-\alpha_n)\|x_n - p\|^2 + \alpha_n \|T^n y_n - p\|^2 + c_n^2 \|T^n y_n - S^n x_n\|^2 \le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n k_n^2 \|y_n - p\|^2 + \alpha_n k \|y_n - T^n y_n\|^2 + \alpha_n \|T^n y_n - S^n x_n\|^2$ …(2.29)

Since S,T are uniformly continuous on the bounded set K, there exists a positive real number $M_1 < \infty$ such that

$\|y_n - T^n y_n\|^2 \le M_1$ and $\|T^n y_n - S^n x_n\|^2 \le M_1$. Therefore (2.29) yields $\|x_{n+1} - p\|^2 \le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n k_n^2 \|y_n - p\|^2 + \alpha_n k M_1 + \alpha_n M_1$ …(2.30)We also have the following estimates, $\|y_n - p\|^2 = \|a_n' x_n + b_n' S^n z_n + c_n' v_n - p\|^2 = \|(1-\beta_n)(x_n - p) + \beta_n (S^n z_n - p) - c_n' (S^n z_n - v_n)\|^2$ $\|y_n - p\|^2 \le (1-\beta_n)\|(x_n - p) - c_n' (S^n z_n - v_n)\|^2 + \beta_n \|(S^n z_n - p) - c_n' (S^n z_n - v_n)\|^2 - \beta_n (1-\beta_n)\|(S^n z_n - p) - (x_n - p)\|^2$ Since $\beta_n (1-\beta_n) \ge 0$, expanding further $\|y_n - p\|^2 \le (1-\beta_n)[\|x_n - p\|^2 + \beta_n^2 \|S^n z_n - v_n\|^2] + \beta_n [\|S^n z_n - p\|^2 + \beta_n^2 \|S^n z_n - v_n\|^2] = (1-\beta_n)\|x_n - p\|^2 + \beta_n \|S^n z_n - p\|^2 + \beta_n^2 \|S^n z_n - v_n\|^2 = (1-\beta_n)\|x_n - p\|^2 + \beta_n k_n^2 \|z_n - p\|^2 + \beta_n k \|z_n - S^n z_n\|^2 + \beta_n^2 \|S^n z_n - v_n\|^2$ …(2.31)

Observe that $c_n' \le \beta_n$ and S is asymptotically demi-contractive also continuity of S on K implies that there exist a positive real number $M_2 < \infty$ such that: $\|z_n - S^n z_n\|^2 \le M_2$ and $\|S^n z_n - v_n\|^2 \le M_2$. Then from (2.31) we have, $\|y_n - p\|^2 \le (1-\beta_n)\|x_n - p\|^2 + \beta_n k_n^2 \|z_n - p\|^2 + \beta_n k M_2 + \beta_n M_2$…(2.32) We also have the following estimates:

$\|z_n - p\|^2 = \|a_n'' x_n + b_n'' T^n x_n + c_n'' w_n - p\|^2 = \|(1-\gamma_n)(x_n - p) + \gamma_n (T^n x_n - p) - c_n'' (T^n x_n - w_n)\|^2 \le (1-\gamma_n)\|(x_n - p) - c_n'' (T^n x_n - w_n)\|^2 + \gamma_n \|(T^n x_n - p) - c_n'' (T^n x_n - w_n)\|^2$ Expanding further we have $\|z_n - p\|^2 \le (1-\gamma_n)[\|x_n - p\|^2 + c_n'' \|T^n x_n - w_n\|^2] + \gamma_n [\|T^n x_n - p\|^2 + c_n'' \|T^n x_n - w_n\|^2] = (1-\gamma_n)\|x_n - p\|^2 + \gamma_n \|T^n x_n - p\|^2 + c_n'' \|T^n x_n - w_n\|^2 = (1-\gamma_n)\|x_n - p\|^2 + \gamma_n k_n^2 \|x_n - p\|^2 + \gamma_n k \|x_n - T^n x_n\|^2 + \gamma_n \|T^n x_n - w_n\|^2$

Continuity of T on the bounded set K implies that there exists a positive real number $M_3 < \infty$ such that: $\|x_n - T^n x_n\|^2 \le M_3$ and $\|T^n x_n - w_n\|^2 \le M_3$. We have $\|z_n - p\|^2 \le (1-\gamma_n)\|x_n - p\|^2 + \gamma_n k_n^2 \|x_n - p\|^2 + \gamma_n k M_3 + \gamma_n M_3$…(2.33)Substituting (2.33) into (2.32) yields,

$\|y_n - p\|^2 \le (1-\beta_n)\|x_n - p\|^2 + \beta_n k_n^2 [(1-\gamma_n)\|x_n - p\|^2 + \gamma_n k_n^2 \|x_n - p\|^2 + \gamma_n k M_3 + \gamma_n M_3] + \beta_n k M_2 + \beta_n M_2 = [1 - \beta_n + \beta_n k_n^2 - \beta_n \gamma_n k_n^2 + \beta_n \gamma_n k_n^4]\|x_n - p\|^2 + (\beta_n \gamma_n k_n^2 k + \beta_n \gamma_n k_n^2 + \beta_n k + \beta_n) M_4$ …(2.34) where $M_4 = \max[M_2, M_3]$. Substituting (2.34) into (2.30) we have $\|x_{n+1} - p\|^2 \le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n k_n^2 [\{1 - \beta_n + \beta_n k_n^2 - \beta_n \gamma_n k_n^2 + \beta_n \gamma_n k_n^4\}\|x_n - p\|^2 + (\beta_n \gamma_n k_n^2 k + \beta_n \gamma_n k_n^2 + \beta_n k + \beta_n) M_4] + \alpha_n k M_1 + \alpha_n M_1 = [1 - \alpha_n + \alpha_n k_n^2 - \alpha_n \beta_n k_n^2 + \alpha_n \beta_n k_n^4 - \alpha_n \beta_n \gamma_n k_n^4 + \alpha_n \beta_n \gamma_n k_n^6]\|x_n - p\|^2 + (\alpha_n \beta_n \gamma_n k_n^4 k + \alpha_n \beta_n \gamma_n k_n^4 + \alpha_n \beta_n k_n^2 k + \alpha_n \beta_n k_n^2) M_4 + \alpha_n k M_1 + \alpha_n M_1$. Let $M = \max[M_1, M_4]$ then we have

$\|x_{n+1} - p\|^2 \le [1 - \alpha_n + \alpha_n k_n^2 - \alpha_n \beta_n k_n^2 + \alpha_n \beta_n k_n^4 - \alpha_n \beta_n \gamma_n k_n^4 + \alpha_n \beta_n \gamma_n k_n^6]\|x_n - p\|^2 + \alpha_n [\beta_n \gamma_n k_n^4 k + \beta_n \gamma_n k_n^4 + \beta_n k_n^2 k + \beta_n k_n^2 + k + 1] M$ …(2.35)

Now setting

$\rho_n = \|x_n - p\|^2 (\delta_n)$. $\delta_n = \alpha_n + \alpha_n k_n^2 - \alpha_n \beta_n k_n^2 + \alpha_n \beta_n k_n^4 - \alpha_n \beta_n \gamma_n k_n^4 + \alpha_n \beta_n \gamma_n k_n^6$ $\sigma_n = \alpha_n [\beta_n \gamma_n k_n^4 k + \beta_n \gamma_n k_n^4 + \beta_n k_n^2 k + \beta_n k_n^2 + k + 1] M$ then (2.36) reduces to: $\rho_{n+1} = (1-\delta_n)\rho_n + \sigma_n$. Clearly $0 \le \delta_n \le 1$, $\sum \delta_n = \infty$ and $\sigma_n = o(\delta_n)$. Hence by lemma 2.2 $\rho_{n \to 0}$ as $n \to \infty$. This implies that $\{\delta_n\}$converges strongly to p the common fixed point of S and T . This complete the proof.

Theorem 2.7: Let B,K,T and the sequence $\{x_n\}$ be as define in theorem 2.6 suppose $S:K \to K$ is nonexpansive. Replace the condition (3) on the parameters replaced with $\sum \alpha_n \beta_n \gamma_n = \infty$. And suppose all other condition are satisfied. If S,T have a common fixed point in K then the sequence $\{x_n\}$ converges strongly to the common fixed point of T and S.

Proof: Since K is uniformly convex and S is nonexpansive the F(S) is nonempty. Also T is asymptotically demi-contractive implies that F(T) is nonempty. Let x* be the common fixed point of S and T. From our hypothesis and by Lemma 2.1 we have the following estimates, $\|x_{n+1} - x^*\|^2 = \|a_n x_n + b_n T^n y_n + c_n S^n x_n - x^*\|^2 = \|(1-\alpha_n)(x_n - x^*) + \alpha_n (T^n y_n - x^*) - c_n (T^n$

$y_n - S^n x_n) \|^2 \le (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n \|T^n y_n - x^*\|^2 + c_n^2 \|T^n y_n - S^n x_n\|^2$ But T is asymptotically demi-contractive and $c_n^2 \le \alpha_n^2$. Therefore $\|x_{n+1} - x^*\|^2 \le (1-\alpha_n)\|x_n - x^*\|^2 + k_n^2\alpha_n \|y_n - x^*\|^2 + k\alpha_n \|y_n - T^n y_n\|^2 + \alpha_n^2 \|T^n y_n - S^n x_n\|^2 = (1-\alpha_n)\|x_n - x^*\|^2 + k_n^2\alpha_n \|y_n - x^*\|^2 + k\alpha_n \|y_n - T^n y_n\|^2 + \alpha_n^2 \|(T^n y_n - x^*) - (S^n x_n - x^*)\|^2 \le (1-\alpha_n)\|x_n - x^*\|^2 + k_n^2\alpha_n \|y_n - x^*\|^2 + k\alpha_n \|y_n - T^n y_n\|^2 + \alpha_n^2 \|T^n y_n - x^*\|^2 + \alpha_n^2 \|S^n x_n - x^*\|^2$ Since T is asymptotically demi-contractive and S is nonexpansive we have $\|x_{n+1} - x^*\|^2 \le (1-\alpha_n)\|x_n - x^*\|^2 + k_n^2\alpha_n \|y_n - x^*\|^2 + k\alpha_n \|y_n - T^n y_n\|^2 + \alpha_n^2 \|y_n - x^*\|^2 + \alpha_n^2 \|y_n - T^n y_n\|^2 + \alpha_n^2 \|x_n - x^*\|^2 = [1-\alpha_n(1-\alpha_n)]\|x_n - x^*\|^2 + \alpha_n (k_n^2 + \alpha_n)\|y_n - x^*\|^2 + \alpha_n(k+\alpha_n)\|y_n - T^n y_n\|^2$ …(2.36) Estimate (2.31) is also valid here. Observing that $c_n \le \beta_n$ we have $\|y_n - x^*\|^2 \le (1-\beta_n)\|x_n - x^*\|^2 + \beta_n k_n^2 \|z_n - x^*\|^2 + \beta_n k\|z_n - S^n z_n\|^2 + \beta_n \|S^n z_n - v_n\|^2$ …(2.37) Also continuity of S on the bounded set K implies that there exists a real number

$R_2 < \infty$ such that : $\|S^n z_n - v_n\|^2 \le R_2$, Substituting into (2.38) yields $\|y_n - x^*\|^2 \le (1-\beta_n)\|x_n - x^*\|^2 + \beta_n k_n^2 \|z_n - x^*\|^2 + \beta_n k \|z_n - S^n z_n\|^2 + \beta_n R_2$ …(2.38) Substituting the equation (2.33) i.e. $\|z_n - x^*\|^2 \le (1-\gamma_n)\|x_n - x^*\|^2 + \gamma_n k_n^2 \|x_n - x^*\|^2 + \gamma_n k M_3 + \gamma_n M_3$ (which also holds in this case) into (2.39) we obtain $\|y_n - x^*\|^2 \le (1-\beta_n)\|x_n - x^*\|^2 + \beta_n k_n^2 [(1-\gamma_n)\|x_n - x^*\|^2 + \gamma_n k_n^2 \|x_n - x^*\|^2 + \gamma_n k M_3 + \gamma_n M_3] + \beta_n k\|z_n - S^n z_n\|^2 + \beta_n R_2 \le [(1-\beta_n) + \beta_n k_n^2(1-\gamma_n) + \beta_n \gamma_n k_n^4]\|x_n - x^*\|^2 + \beta_n \gamma_n k_n^4 k M_3 + k_n^2 \gamma_n M_3 + \beta_n k\|z_n - S^n z_n\|^2 + \beta_n R_2 \le [1 - \beta_n + \beta_n k_n^2 - \beta_n \gamma_n k_n^2 + \beta_n \gamma_n k_n^4]\|x_n - x^*\|^2 + [\beta_n \gamma_n k_n^2 k + \gamma_n k_n^2 + \beta_n] R_4 + \beta_n k\|z_n - S^n z_n\|^2$ …(2.39) where $R_4 < \infty$ is a real number such that: $R_4 = \max[M_3, R_2]$.

Substituting (2.39) into (2.36) and observing that T is continuous on the bounded set K implies that there exists a positive real number $R_5 < \infty$ such that:

$\|y_n - T^n y_n\|^2 \le R_5$, we have $\|x_{n+1} - x^*\|^2 \le [1 - \alpha_n(1-\alpha_n)]\|x_n - x^*\|^2 + \alpha_n(k_n^2 + \alpha_n)[\{1 - \beta_n + \beta_n k_n^2 - \beta_n \gamma_n k_n^2 + \beta_n \gamma_n k_n^4\}\|x_n - x^*\|^2 + \{\beta_n \gamma_n k_n^2 k + \gamma_n k_n^2 + \beta_n\} R_4 + \beta_n k\|z_n - S^n z_n\|^2] + \alpha_n(k+\alpha_n)R_5 = [1 - \alpha_n(1-\alpha_n) + \alpha_n(k_n^2 + \alpha_n)(1 - \beta_n + \beta_n k_n^2 - \beta_n \gamma_n k_n^2 + \beta_n \gamma_n k_n^4)]\|x_n - x^*\|^2 + [\beta_n \gamma_n k_n^2 k + k_n^2 \gamma_n + \beta_n + \alpha_n(k+\alpha_n)] R_6 + \beta_n k\|z_n - S^n z_n\|^2$ where $R_6 < \infty$ is a real number such that: $R_4 = \max[R_4, R_5]$. Now set $\phi_n = \|x_n - x^*\|^2$ $\delta_n = \alpha_n(1-\alpha_n) + \alpha_n(k_n^2 + \alpha_n)(1 - \beta_n + \beta_n k_n^2 - \beta_n \gamma_n k_n^2 + \beta_n \gamma_n k_n^4)$ $\sigma_n = [\beta_n \gamma_n k_n^2 k + k_n^2 \gamma_n + \beta_n + \alpha_n(k+\alpha_n)] R_6 + \beta_n k\|z_n - S^n z_n\|^2$ Then (2.39) reduces $\phi_{n+1} = (1-\delta_n)\phi_n + \sigma_n$. Observe that $0 \le \delta_n \le 1$ and $\sigma_n = o(\delta_n)$. Also $\sum \delta_n = \infty$ from hypothesis. Hence by lemma 2.2, $\phi_{n \to 0}$ as $n \to \infty$. This implies that $\{x_n\}$ converges strongly to x^* the common fixed point of S and T. This complete the proof.

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